THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

Vol. IX. | December, 1917. | No. 132.

GEOMETRY AT INFINITY.

By Professor D. M. Y. Sommerville, D.Sc.

Mr. Hatton in the March number of the *Gazette* (p. 43) has given an interesting discussion of the "geometry at infinity" with the object of reconciling Euclid's definition of parallel straight lines with Desargues' conception of lines "meeting at infinity." There seems to me, however, still to be considerable confusion, and the young geometrical student who finds difficulty in grasping the statement that all the points at infinity in a plane lie on one straight line will scarcely be reassured when he is informed that two planes may under certain circumstances intersect in a region which is neither a straight line nor indeed a curve.

It is a somewhat curious paradox, in this region of paradoxes, that while, historically, points at infinity have been introduced by projective geometry, projective geometry, properly speaking, has nothing to do with points at infinity, but completely ignores them. The explanation is found to lie in the intermediate stage of development. There are three stages in the evolution of projective geometry:

(1) The older Metrical Geometry, or the Geometry of Euclid, in which there is no idea of points at infinity, but which brims with exceptional cases always requiring separate statements.

(2) The method of projection and the principle of continuity applied to metrical geometry (associated with the name of Poncelet). It is here that "points at infinity" are introduced in order to remove the exceptional cases; for example, in order to be able to say that a straight line is always projected into a straight line, the idea of "line at infinity" is invented to correspond to the "vanishing line" in the projected plane.

(3) Pure Projective Geometry (Von Staudt), in which the scaffolding of metrical geometry is discarded, leaving a completely homogeneous system in which two planes, without exception, intersect in a straight line.

It is at stage (2) then, and only at this stage, that points at infinity find a place, and the confusion and paradoxes that exist here are due entirely to the practice of using one and the same term with different meanings. There is an absolute contradiction between Euclid's state-

ment that parallel straight lines do not intersect and Desargues' that they intersect at infinity. It is no better when we try to picture the point of intersection at infinity as the limiting case of ordinary intersection. There are two ways, not quite distinct, in which the idea of the point at infinity on a straight line introduces itself.

The first is by considering the sequence of points $P_1, P_2, \ldots$ corresponding to segments $OP=1, 2, \ldots$ measured from a fixed point O on the line. The limiting position of P as the segment $OP \to \infty$ is called the point at infinity on the line. But there is no such limiting point any more than there is a definite number ∞.

The other method is by allowing the line AP to rotate about a fixed point A while its point of intersection P with a fixed line OX moves along OX. As the segment $OP \to \infty$, the line AP tends to a definite position certainly, but this limiting line does not belong to the system of lines formed by joining A to points of OX, and the limiting position of P again evades us.

The following method of developing the geometry at infinity from the axioms of metrical geometry contains nothing new, but it may be found to be of interest as the point of view is rather different from that of the usual text-books. For shortness the discussion is confined to plane geometry, but it can on similar lines be modified and extended to three dimensions.

We begin with Hilbert's "axioms of connection."

The *point* and *line* are taken as undefined.

1. *Two distinct points uniquely determine a line.*
2. *Any two distinct points of a line determine that line.*
3. *On a line there are always at least two points.*

3a. *There are at least three points not in the same line.*

From 1 it follows that two lines have either *one* or *no* point in common.

Axiom of Parallels. *If a is any line and A a point not in a, then there is one and only one line b which passes through A and does not cut a.*

Def. 1. The line b is said to be parallel to a.

The reciprocal of 1 is:

A. *Two distinct lines uniquely determine a point;*

but this is not true. It is our object to modify the definitions so that both 1 and A may be universally true. For this purpose we introduce new definitions as follows:

Def. 2. An "actual point" consists of a system of concurrent lines (pencil). [There is nothing really startling in this definition, since the reciprocal definition, "a line consists of a system of collinear points," is commonplace.]

Def. 3. A "point at infinity" consists of a system of parallel lines. [It is easily proved from Def. 1 that if $a \parallel c$ and $b \parallel c$, then $a \parallel b$, so that we can speak of a system of parallel lines.]

Def. 4. A "point" is either an "actual point" or a "point at infinity."

With these definitions the modified form of A,

A′. *Two distinct lines uniquely determine a "point,"*

is now universally true.

But we have now to examine 1 for the three cases, two "actual points," an "actual point" and a "point at infinity," two "points at infinity."

Two "actual points" uniquely determine a line, viz. the line which is common to the two pencils.

An "actual point" (pencil with vertex A) and a "point at infinity" (lines $\parallel b$) uniquely determine a line, viz. the line through $A \parallel b$.

Two "points at infinity," however, do not determine a line, since by Def. 1 two distinct and not parallel lines cannot both be parallel to a third line.

We must therefore extend the definition of a line, and we say that in this case the two "points at infinity" determine a "line at infinity," and that a "line" is either an "actual line" or "line at infinity."

Some theorems on the "line at infinity" can now be obtained.

Th. 1. An "actual line" contains one and only one "point at infinity." [All the lines $\parallel a$ determine a "point at infinity" which belongs to a, but if there were two such "points at infinity," a would be a "line at infinity."]

Th. 2. A "line at infinity" contains no "actual points." [For an "actual point" determines with a "point at infinity" an "actual line."]

With these extensions the modified form of 1,

1′. *Two distinct "points" uniquely determine a "line,"*

is now universally true.

But we have now to examine A′ once more for the three cases, two "actual lines," an "actual line" and a "line at infinity," two "lines at infinity."

The first case has been already examined and found true.

An "actual line" and a "line at infinity" have to determine a unique "point," but this can only be the unique "point at infinity" which lies on the "actual line." Let l_∞ be some particular "line at infinity" and P_∞ any "point at infinity." P_∞ determines with any "actual point" an "actual line" a. But we have just seen that the "point at infinity" on a must lie on l_∞. Hence l_∞ contains all the "points at infinity." Hence there is only one "line at infinity"; the third case is therefore non-existent, and the modified form of A,

A″. *Two distinct "lines" uniquely determine a "point,"*

is now universally true. D. M. Y. SOMMERVILLE.

Victoria Univ. Coll.,
Wellington, N.Z., 31st July, 1917.

---

7. Sublime. "It may be worth while to state in a few words and to prevent a reader of the older mathematicians from imagining that they spoke rhapsodically, that the term *sublime* geometry was technical, meaning the higher parts of geometry, in which the infinitesimal calculus or something equivalent was employed."—De Morgan [*Sublime*, P.C. 1842].

8. Rhapsodists. "(Vieta) states that he gets his denominations from certain *Raphsodi* (as he calls them; it is not often that mathematical tabulators are called rhapsodists) whom he does not name."—De Morgan [*Tables*, P.C. Supp. 29, p. 597].

9. Cascades. "Rolle's treatise on algebra (1690) contains, among other methods for the solution of equations, one which he calls the method of *cascades*, a name given to it because it consists in successively depressing the equation one degree lower at each operation. It has some analogy to a method given by Newton in his 'Arithmetica Universalis'; but its want of generality has caused it ever since to be neglected."—De Morgan [*Rolle*, P.C. 1841].

MATHEMATICAL NOTES.

519. [V. 2.] *Gazette, No.* 131, *p.* 149, *l.* 21.

The lines are from Tennyson's *Lucretius*, 106-110.—*The Reviewer.*

520. [V. 1.] *The early stages in Elementary Algebra.*

The following note is intended for mathematical masters who have the opportunity of making experiments, and are not afraid to make them. The writer experimented with private pupils, and extended to classes the experience thereby acquired. By linking together the ordinary processes in a fashion somewhat as follows, he found that the ground could be covered more rapidly and effectively than is usually the case. No other method in his experience gives so close a grip of the fundamental ideas in factorising and leads so swiftly to the recognition of the essential points in the theory of quadratic equations. Most of the work was done *vivâ voce* and at the blackboard. Apart from the gain of rapidity in performing the ordinary operations, the faculty of working "by inspection" in itself implies a grasp of principles. The presence of that grasp cannot always be detected with certainty in those who are taught in separate departments certain processes, apparently unconnected, but really in close relation.

1. There is no need to dwell on simple addition or subtraction. Where there is no complication of brackets, the student selects the corresponding terms, performs the required operation, and says aloud, or writes, "By inspection, the sum (or difference) is . . ."

2. Next teach the distributive law. The pupil realises without difficulty that in the expansion of the form $(px \pm q)(ax \pm b)$ the terms consist of a descending order of powers of x beginning with x^2 and ending with what may be abbreviated into "no x." Similarly, with $(px \pm q)(ax^2 \pm bx \pm c)$, we have in turn, x^3, x^2, x, and "no x," and so on. After drill in the number of ways in which a term may be formed, *e.g.* $x^2 = x^2 \times$ "no x," or $x^2 \times x$, etc., we can go on to forms $(ax^2 \pm bx \pm c)(px^2 \pm qx \pm r)$, and the like.

In all these cases the work is purely mental. All that is said aloud, or written down is "By inspection the product is . . ."

It is usual to begin with forming products of the type $(x \pm 3)(x \pm 5)$, and later of the form $(3x \pm 5)(5x \pm 6)$. I believe that there is no gain in this. If we first take the general forms, it prevents the "compartment" idea from arising; the process is just as readily grasped, and no time is wasted. Two invaluable habits are inculcated throughout this stage: the keeping of a watchful eye upon dimensions, and the habit of "checking" at every step.

After each expansion of forms like $(x+5)(x+10)$ into $x^2+15x+50$, we may discuss the values of x that make each factor vanish, and we substitute those values in the $x^2+15x+10$, thus driving home the lesson to be learned in the "remainder theorem." Now or later we may deal with the solution of $x^2+15x+50=0$.

3. Having discovered such relations as $(3x-1)(2x-5)=6x^2-17x+5$, we reverse our steps and at once find the type: What factors will give us $6x^2-17x+5$? I claim that the previous formation of the product by inspection affords us the surest, clearest, and swiftest insight into the method by which the factors in their turn may be discovered. There are no rules to be learned. The reasons for everything leap to the eyes.

4. Now, or earlier, it is natural to reverse the process: Given that one factor of $6x^2-17x+5$ is $2x-5$, find the other, or what is the same thing, divide $6x^2-17x+5$ by $2x-5$.

The first step is obvious: $6x^2-17x+5=(2x-5)(3x \quad)$.

We see on distributing as far as we can that we have $-15x$, and that we want $-17x$, so that $-2x$ more is required, of which we already have as factor $+2x$. Hence our missing term must be this remaining factor -1. We say "must be," because no term can come between an "x" and a "no x."

Hence $$6x^2-17x+5=(2x-5)(3x-1),$$

so that $3x-1$ is the quotient or factor required.

Similarly, we can deal with the type: Divide $6x^2-17x+51$ by $2x-5$.

We have at once, as above,

$$6x^2-17x+51=(2x-5)(3x-1)\ldots.$$

But while in the last case we had $(-5)(-1)=+5$, which was required, we have now $+5$ and want 51, *i.e.* an additional 46.

Hence $$6x^2-17x+51=(2x-5)(3x-1)+46,$$

i.e. $3x-1$ is the quotient and $+46$ is the remainder. And we note result for $x=2\frac{1}{2}$ or $\frac{1}{3}$.

Again: Divide $6x^2-20x+51$ by $2x-5$.

We have at sight $\quad 6x^2-20x+51=(2x-5)(3x \qquad)$.

We have $-15x$ and want $-20x$, and want $-5x$ more. One factor being $+2x$, we must consider the $-5x$ as $-4x-x$, so that the missing term is -2, and we have $-x$ remaining. Thus

$$6x^2-20x+51=(2x-5)(3x-2)-x\ldots.$$

Finally, we have $(-5)(-2)=+10$, and we want $+51$, giving a remainder of 41.

Thus $$6x^2-20x+51=(2x-5)(3x-2)-x+41.$$

And $3x-2$ is the quotient, and $41-x$ is the remainder.

We also see the condition that $2x-5$ may be a factor of the dividend, viz. $x=41$, and that the same condition is necessary if $3x-2$ is a factor.

5. Noting that $6x^2-17x+5=6(x^2-\frac{17}{6}x+\frac{5}{6})$, and that

$$(2x-5)(3x-1)=2(x-\tfrac{5}{2})3(x-\tfrac{1}{3})=6(x-\tfrac{5}{2})(x-\tfrac{1}{3}),$$

or in the simpler case $x^2+15x+50=(x+5)(x+10)$, the student can *see* in the equations $x^2+15x+50=0$, $6x^2-17x+5=0$

why the sum of the roots is -15, and *why* it is negative;
why the product of the roots is 50, and *why* it is not -50;
why the sum of the roots is $\frac{17}{6}$, and *why* it is not negative;
why the product of the roots is $\frac{5}{6}$, and *why* it is not $-\frac{5}{6}$;

and *why*, when the two roots of a quadratic are given, the equation may be written at once in the form, say, $(x+a)(x+b)=0$, and so on.

6. It is easy to extend our division to "more difficult cases."

Divide $x^3-x^2-9x-12$ by x^2+3x+3.

Here an appeal to dimensions shows that the quotient is a linear factor; and as $x^3\div x=x^2$ and $(-12)\div(+3)=-4$, the quotient is probably $x-4$.

Writing $x^3-x^2-9x-12=(x^2+3x+3)(x-4)$, and checking the terms not verified, we see that there is no remainder.

Divide $14x^4+45x^3y+78x^2y^2+45xy^3+14y^4$ by $2x^2+5xy+7y^2$; we have at once

$$14x^4+45x^3y+78x^2y^2+45xy^3+14y^4=(2x^2+5xy+7y^2)(7x^2 \qquad +2y^2).$$

Now we have $35x^3y$ and want $45x^3y$; therefore $10x^3y$ more is wanted, of which we have one factor $2x^2$, so that the missing term is $+5xy$.

Thus

$$14x^4+45x^3y+78x^2y^2+45xy^3+14y^4=(2x^2+5xy+7y^2)(7x^2+5xy+2y^2)\ldots,$$

and on checking for the terms involving x^2y^2 and xy^3 we see that there is no remainder.

Again: Divide $x^{12}-y^{12}$ by x^2-y^2.

Here $$x^{12}-y^{12}=(x^2-y^2)(x^{10} \qquad +y^{10}).$$

We have $-x^{10}y^2$ and we want no term involving $x^{10}y^2$, so that $+x^{10}y^2$ is required, of which we have one factor x^2; hence the next term is $+x^8y^2$, and so on, to

$$x^{12}-y^{12}=(x^2-y^2)(x^{10}+x^8y^2+x^6y^4+x^4y^6+x^2y^8 \qquad).$$

We have $-x^2y^{10}$ and want "none," so that $+x^2y^{10}$ is needed, of which we have x^2; thus the next term is $+y^{10}$, and

$$x^{12}-y^{12}=(x^2-y^2)(x^{10}+x^8y^2+\ldots+x^2y^8+y^{10}).$$

Lastly, $y^{10}(-y^2)=-y^{12}$ as required. Check the "weights"

$$10=8+2=6+4=\ldots.$$

7. By inspection, the student sees that

$$\begin{aligned}(a+b+c+\dots)^2 = \Sigma a^2 &+ 2a \text{ (all terms that follow)}\\ &+2b\ (\dots\dots\dots\dots\dots\dots\dots\dots)\\ &+2c\ (\dots\dots\dots\dots\dots\dots\dots\dots)\\ &+\dots\ .\end{aligned}$$

Hence square roots yield to inspection.

Take the "difficult" example: Find the 4th root of

$$E \equiv 1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8.$$

Here $\qquad E=(1 \qquad\qquad x^4)^2.$

The $8x$ is $2\times 1\times$ next term; $\therefore$ next term is $4x$, and

$$E \equiv (1+4x \qquad\qquad x^4)^2.$$

The $20x^2$ is $(4x)^2+2\times 1\times$ next term; $\therefore$ next term is $2x^2$, and

$$E \equiv (1+4x+2x^2 \qquad\qquad x^4)^2.$$

The $-8x^7=2\times x^4\times$ last term but one; $\therefore$ last term but one is $-4x^3$, and

$$\begin{aligned}E &= (1+4x+2x^2-4x^3+x^4)^2\\ &= (1 \qquad\qquad\qquad x^2)^4.\end{aligned}$$

The $4x$ is $2\times 1\times$ second term, so that

$$E=(1+2x \qquad\qquad x^2)^4.$$

There is no other term, for no term lies between an x and an x^2.

Then check each term so far untested. Thus $-4x^3=2x^2\times$ last term but one, so that sign of x^2 is negative. Common sense also tells that the x^2 must be $-x^2$, or, on raising to the 4th power, all terms would be positive.

8. As a rule the ordinary cube roots are found at once by noting that

$$(a+b+\dots)^3=\Sigma a^3+3\Sigma a^2b+\dots\ .$$

So that we can write at once that

$$1+6x+15x^2+20x^3+15x^4+6x^5+x^6=(1+2x+x^2)^3.$$

For $6x$ is $1\times 3\times$ 2nd term, and the cube root of $+x^6$ is $+x^2$.

Care must be taken to check all the untested terms to make sure that the expressions are perfect squares or perfect cubes.

9. Find the H.C.F. of

$$x^4+2x^3-3x^2-4x-1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(A)$$

and

$$x^5+3x^4+3x^3+7x^2+5x+1. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(B)$$

If they have a common factor it is a factor of their sum, viz. of

$$x^5+4x^4+5x^3+4x^2+x=x(x^4+4x^3+5x^2+4x+1)=Cx \text{ (say)}.$$

Rejecting x, for it is not a factor in A and B, the C.F. required is a factor of $A+C$, *i.e.* of $2x^4+6x^3+2x^2=2x^2(x^2+3x+1)$.

Rejecting $2x^2$ as not a factor of A or B, we have for the only possible common factors $\qquad x^2+3x+1$, its factors, or unity. ....(α)

Trial shows that

$$x^4+2x^3-3x^2-4x-1=(x^2+3x+1)(x^2\dots-1).$$

We have $3x^3$ and want $2x^3$; thus $-x^3$ is required, of which we have x^2, so that missing term is $-x$.

Hence $\qquad x^4+2x^3-3x^2-4x-1=(x^2+3x+1)(x^2-x-1).$ ....(β)

[Checking for x^2 on the R.H. we have $-3+1-1=-3$,

.... x .... $-1-3=-4$.]

So also $\qquad x^5+3x^4+3x^3+7x^2+5x+1=(x^2+3x+1)(x^3 \qquad\qquad +1).$

We have $3x^4$ and want $3x^4$. Hence no other term in x^4 is required, so there is no x^2 in the second bracket.

We have x^2 and want $3x^2$, so that $2x^2$ is required, and as x^2 is one factor of it, the missing term is $+2x$.

Hence $\qquad x^5+3x^4+3x^3+7x^2+5x+1=(x^2+3x+1)(x^3+2x+1).$ ....(γ)

So that $\qquad x^2+3x+1$ is the H.C.F. of A and B, ...(δ)

and $\qquad (x^2+3x+1)(x^3+2x+1)(x^2-x-1)$ is the L.C.M. of A and B. ....(ϵ)

Thus the whole of the work to be written down after the line numbered (α) consists of the lines numbered (β), (γ) and (δ) or (ϵ).

No apology is needed, I hope, for giving at length the work of determining by inspection the coefficients of the successive terms, for I understand that the Editor has some difficulty in securing notes of an elementary type.

My classes consisted of boys of very ordinary capacity, and I found that the working out of products, quotients, roots, etc., presented no difficulty whatever. The gain in time alone is considerable enough to justify a trial by others, with such modifications, extensions, or alterations in order, as may seem appropriate to the individual teacher in the course of his experiment. Gain of time, however, is by no means the only gain.

EXPERTO CREDE.

521. [K[1]. 2. e.] *Note on the Distances of the In-centre and Ex-centres of a Triangle from any Vertex.*

In Hobson's *Trigonometry*, p. 191, the following Problem is proposed :

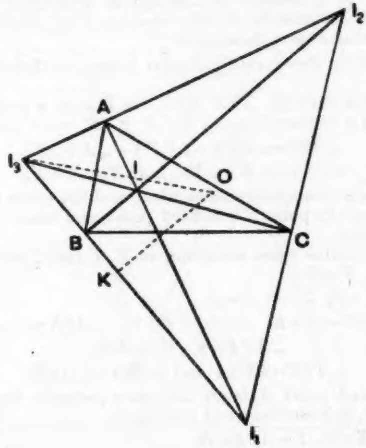
If a, a_1, a_2, a_3 are the distance of the centres of the inscribed and escribed circles from A, and p is the perpendicular from A on BC, prove that

(1) $aa_1a_2a_3=4R^2p^2$.

(2) $a^2+a_1^2+a_2^2+a_3^2=16R^2$.

(3) $a^{-2}+a_1^{-2}+a_2^{-2}+a_3^{-2}=4p^{-2}$.

I suggest the following solution as interesting, with usual lettering :



(1) $$\angle I_3AB=\angle I_2AC,$$

and since $AIBI_3$ and $AICI_2$ are cyclic quadrilaterals,

$$\angle AI_3B=\angle AII_2=\angle ACI_2;$$

$\therefore$ the triangles ABI_3, AI_2C are similar ;

$$\therefore\ AI_3 : AB=AC : AI_2;$$

$$\therefore\ a_2a_3=bc.$$

Again, $$\angle BAI=\angle IAC,$$

and since ABI_1I_2 is a cyclic quadrilateral,

$$\angle ABI=\angle AI_1C;$$

$\therefore$ the triangles ABI, AI_1C are similar;

$$\therefore\ AI : AB = AC : AI_1;$$

$$\therefore\ aa_1 = bc.$$

Hence $$aa_1a_2a_3 = b^2c^2 = 4R^2p^2.$$

(2) $$a^2+a_1^2+a_2^2+a_3^2=(a^2+a_2^2)+(a_1^2+a_3^2)$$
$$=II_2^2+II_3^2$$
$$=4OK^2+4I_3K^2$$

(O is the circumcentre of $I_1I_2I_3$ and OK perpendicular to I_1I_2)

$$=4OI_3^2;\ \text{but } OI_3=2R;$$

$$\therefore\ a^2+a_1^2+a_2^2+a_3^2=16R^2.$$

(3) $$a^{-2}+a_1^{-2}+a_2^{-2}+a_3^{-2}=\frac{a^2+a_1^2}{a^2a_1^2}+\frac{a_2^2+a_3^2}{a_2^2a_3^2}$$
$$=\frac{a^2+a_1^2+a_2^2+a_3^2}{4R^2p^2}\ \text{(by (1))}$$
$$=\frac{16R^2}{4R^2p^2}=4p^{-2}.$$

C. H. Richard.

522. [K[1]. 5. d.] *If a triangle is inscribed in a directly similar triangle, and if one triangle varies while the other remains fixed, the centre of similitude is either a fixed point or on a fixed circle.*

Let ABC, XYZ be two directly similar triangles, X being on BC, Y on CA, Z on AB.

The three circles through AYZ, BZX, CXY have a common point S, at which YZ, ZX, XY subtend angles $\pi-A$, $\pi-B$, $\pi-C$ respectively. Also

$$\angle BSC=\angle BZX+\angle CYX=\angle A+\angle X,$$

and similarly $$\angle CSA=\angle B+\angle Y,\quad \angle ASB=\angle C+\angle Z.$$

Hence, if one of the triangles remains fixed, and the other varies, remaining constant in species, the point S is a fixed point, and bears a fixed relation to each of the triangles.

There are six possible cases according as X, Y and Z are or are not equal respectively to A, B and C.

Case I. Let $X=B$, $Y=C$, $Z=A$.

Then $$\angle BSC=A+B,\quad \angle CSA=B+C,\quad \angle ASB=C+A.$$

Also $$\angle XSY=\pi-C=\angle BSC,$$

and similarly $$\angle YSZ=\angle CSA \text{ and } \angle ZSX=\angle ASB.$$

Therefore the fixed point S bears the same relation to the two similar triangles: that is, it is the centre of similitude.

Case II. Let $X=C$, $Y=A$, $Z=B$.

Then, as in Case I., S is the centre of similitude of the two triangles.

It is the point at which BC, CA, AB subtend angles $A+C$, $B+A$, $C+B$ respectively.

For these two cases, see Casey's *Sequel to Euclid*, supplementary chapter, section 2. The two positions of S are the Brocard Points of either triangle.

Case III. Let $X=A$, $Y=B$, $Z=C$.

Then $\angle YSZ=\pi-A=\pi-X$, and $\angle ZSX=\pi-Y$, $\angle XSY=\pi-Z$;

$$\therefore\ S \text{ is the orthocentre of } XYZ.$$

Also $$\angle BSC=A+X=2A,\ \text{and } \angle CSA=2B,\ \angle ASB=2C;$$

$$\therefore\ S \text{ is the circumcentre of } ABC.$$

Suppose ABC to remain fixed and XYZ to vary. Then, since XYZ is of constant species and has one point, the orthocentre, fixed, and since X moves along a straight line, the locus of any other point bearing a fixed relation to the triangle is a straight line. In particular, the locus of the centroid H is a straight line. To determine the position of this straight line, let X, Y, Z coincide with D, E, F, the middle points of the sides of the triangle ABC. Then the centroid coincides with G, the centroid of ABC. Therefore S, G, D, three points connected with the triangle DEF, correspond to the points S, H, X in the similar triangle XYZ. $\therefore$ SDX and SGH are similar triangles. $\therefore$ $\angle SGH$ is a right angle ; *i.e.* the locus of H is the straight line through G, perpendicular to SG.

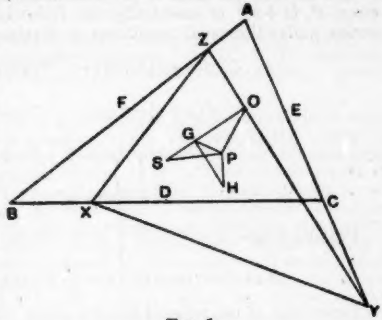


Fig. 1.

Let O be the orthocentre of ABC, and P the double point.

Then since O and S are a pair of corresponding points, and G and H another pair, the triangles SPO and HPG are similar, and $\angle PGH = \angle POS$. $\therefore$ GH touches the circle circumscribing POG : *i.e.*, P lies on the circle on OG as diameter, the orthocentroidal circle of the fixed triangle. From the definition of the double point, it must also be on the orthocentroidal circle of XYZ, which will therefore be the locus if ABC varies and XYZ remains fixed.

Case IV. Let $X=A$, $Y=C$, $Z=B$.

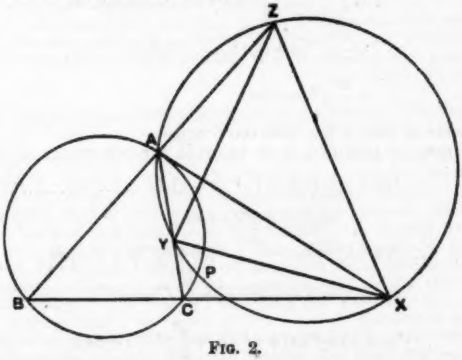


Fig. 2.

In this case X is a fixed point, namely the point where the tangent at A to the circumcircle of ABC meets BC produced. The circumcircles of ABC, XYZ evidently intersect in A : let P be their other point of intersection. Then the sides of the triangle XYZ subtend at P the same angles as the corresponding sides of ABC. P is therefore the double point of the two triangles : and the locus of the double point is the circumcircle of the fixed triangle.

Similarly the locus will evidently be the circumcircle of the fixed triangle in

Case V., where $Y=B$, $Z=A$, $X=C$, and in

Case VI., where $Z=C$, $X=B$, $Y=A$.

T. J. Richards.

523. [C. 1. a.] *Note on Taylor's Series.*

The method given in a previous note for obtaining expansions of $\sin x$, $\cos x$, e^x, $(1+x)^n$ is essentially the following, by means of which Taylor's series, under the usual conditions, is obtained with great ease.

$$f(a+h)=f(a)+\int_0^h f'(a+u)du,$$

$$f'(a+u)=f'(a)+\int_0^u f''(a+u)du\,;$$

$$\therefore\ f(a+h)=f(a)+\int_0^h\left[f'(a)+\int_0^u f''(a+u)\right]du$$

$$=f(a)+hf'(a)+\int_0^h\int_0^u f''(a+u)du^2.$$

Putting $f''(a+u)=f''(a)+\int_0^u f'''(a+u)du$, and integrating,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\int_0^h\int_0^u\int_0^u f'''(a+u)du^3.$$

Repetition of the process plainly gives

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+\int_0^h\int_0^u\ldots\int_0^u f^n(a+u)du^n.$$

The integral is equal to $\int_0^h\int_0^u\ldots\int_0^u uf^n(a+\theta_1 u)du^{n-1}$

$$=\int_0^h\int_0^u\ldots\int_0^u\frac{u^2}{2!}f^n(a+\theta_2\theta_1 u)du^{n-2}$$

$$=\int_0^h\frac{a^{n-1}}{(n-1)!}f^n(a+\theta_{n-1}\ldots\theta_2\theta_1 u)du$$

$$=\frac{h^n}{n!}f^n(a+\theta_n\,.\,\theta_{n-1}\ldots\theta_2\theta_1 u)$$

$$=\frac{h^n}{n!}f^n(a+\theta u),$$

where the value of each θ lies between 0 and 1.

The proof may be modified so as to avoid the occurrence of the repeated integral.

$$f(a+h)=f(a)+\int_0^h f'(a+u)du \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$=f(a)+hf'(a+\theta h).$$

$$\therefore\ f(a+h)=f(a)+\int_0^h[f'(a)+uf''(a+\theta u)]du\,;$$

$$\therefore\ f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h),$$

and $$\therefore\ f'(a+u)=f'(a)+uf''(a)+\frac{u^2}{2!}f'''(a+\theta u),$$

where θ merely denotes a positive proper fraction, not in general the same in different expressions.

Substituting for $f'(a+u)$ in (1) and integrating,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a+\theta h).$$

As before, deduce from this equation the corresponding expansion for $f'(a+u)$ and substitute in (1).

Integrating, we obtain for $f(a+h)$ Taylor's Series with Lagrange's remainder after four terms, and the general result is obvious. C. T. Preece.

REVIEWS.

Finite Collineation Groups: With an Introduction to the Theory of Groups of Operators and Substitution Groups. By H. F. Blichfeldt. Pp. xii, 194. 6s. 6d. net. (Chicago: The University of Chicago Press; Cambridge University Press.)

The theory of finite groups of linear substitutions may be said to have been originated by Klein in 1876 in connection with a problem in the theory of invariants; subsequently he extended Galois's theory of algebraic equations by the introduction of linear groups. From 1873 to 1878 the solutions of Schwarz, Fuchs, and Jordan of an important problem connected with linear differential equations hinged upon the discovery of the invariants of certain corresponding linear groups or the groups themselves, although the notion of group was not at first introduced, except by Jordan. From this time the theory has developed with rapidity, and its development is to be found mainly in scattered articles in mathematical journals, in addition to a few text-books on group theory. In 1916 the present author contributed to the *Theory and Applications of Finite Groups* the part on finite groups of linear homogeneous transformations, which in certain respects is on lines similar to the present work, but in many respects the present work differs from it: the outline of the different principles contained in publications on group theory depends on a minimum of abstract group theory, and there is more of the theory of linear groups. A collineation group may be exhibited as a linear group and *vice versa*, so that it is necessary to discuss only one of these categories, preferably the latter, which lends itself more readily to study (p. 63). The chapters of the book are concerned with the elementary properties of linear groups; groups of operators and substitution groups; linear groups in two variables; advanced theory of linear groups; linear groups in three variables; theory of group characteristics; linear groups in four variables; and some details on the history and applications of linear groups. The book is clearly and well written, and is a most valuable addition to the series of text-books which are published by the Chicago University Press, and which are intermediate between articles in journals and the usual text-books.

Philip E. B. Jourdain.

William Oughtred, a Great Seventeenth-Century Teacher of Mathematics. By F. Cajori. Pp. vi + 100. 4s. net. 1917. (Open Court Company.)

On the seventeenth of August, 1653, Evelyn records that he went to visit Mr. Hyldiard at Horsley, in the house which once was Sir Walter Raleigh's, and there "met me Mr. Oughtred the famous mathematician; he shew'd me a box or golden case of divers rich and aromatic balsams, which a chymist a scholar of his had sent him out of Germany." On August 28th, two years later, we find the further entry: "Came that renown'd mathematician Mr. Oughtred to see me, I sending my coach to bring him to Wotton, being now very aged. Amongst other discourse he told me he thought water to be the philosopher's first matter, and that he was well perswaded of the possibility of their elixir; he believ'd the sunn to be a material fire, the moone a continent, as appears by the late Selenographers; he had strong apprehensions of some extraordinary event to happen the following yeare, from the calculation of coincidence with the diluvian period; and added that it might be possible to convert the Jewes by our Saviour's visible appearance, or to judge the world; and therefore his word was *Parate in occursum*; he said original sin was not met with in the Greeke Fathers, yet he believ'd the thing; this was from some discourse on Dr. Taylor's late booke which I had lent him." *

* What would he have said had he lived to see built in his parish by Croker's friend, Henry Drummond, a church for the Irvingite sect, at a cost of £16,000!—"well attended at the present day" (1884).

Oughtred must have interchanged with his host some painful reflections upon the chances and changes of this mortal life, as the future Bishop of Down and Connor was at this moment languishing, for some innocent or calculated indiscretion, perhaps in this very "late booke," a prisoner within the walls of Chepstow Castle. The agreeable and industrious young philosopher of Wotton and Sayes Court had in that year been so drawn to Jeremy Taylor by community of spiritual vision that he had made him henceforward his "ghostly father." Just at this time Evelyn was engaged upon a study of the "vast argument," to use a phrase of the poet Waller, of the *De Rerum Naturâ*, the first book of which he published in English verse (1656), and the elder poet could hardly have found words more inclusive and conclusive in which to praise the result of the labours of his friend than: "For here Lucretius whole we find, His words, his music, and his mind." * It was natural that in the course of conversation between such a host and such a guest their thoughts should move from the nature of matter to the constitution of the heavenly bodies, and to the deeper subjects which, we may be sure, were occupying the mind of the rector of Albury (or Aldbury, as it used to be written) in the closing years of his life. The vignette above, which we have transferred from pages of the famous diary, is an interesting addition to the scanty store of information possessed with respect to Oughtred apart from the absorbing studies which won for him from his friend and protector the astrologer, William Lilly, the title of the "most famous mathematician of all Europe," and from that distinguished connoisseur in "Worthies," Thomas Fuller (the "stuff and substance of whose intellect, as Coleridge said, was wit"), the title of "prince of mathematicians." It is strange that we have had to wait until now for a monograph dealing with the life and works of one who was so famous among his contemporaries. But a search among the sources of information raises no dust of Simancas, and the little we get is not always to be trusted—*e.g.*, as a glaring instance, the garrulous gossip of old Aubrey. That he was a man of considerable personal magnetism may be gathered from the fact that we know of but one serious quarrel in which he was engaged, and his were days when men fought with the bludgeon, as witness the combat of Dr. John Wallis with the philosopher of Malmesbury. The mention of Hobbes, whose portrait was painted by Wenceslaus Hollar in 1656, reminds us that the *Biog. Brit.* gives *aet.* 73 for the same artist's portrait of Oughtred.† This was subsequently engraved by Hollar and by Faithorne. Mr. W. W. Rouse Ball states in *Science Progress* (April, 1917), that the portrait was first drawn in 1644. Where is the original now? The Earl of Arundel was the patron both of Hollar and of Oughtred, so that if the portrait was in his collection, it should not be difficult to trace. And, as the Open Court Co. has already printed a portrait of Oughtred, why have we not one in this volume?

That Oughtred had countless friends is known from his wide correspondence. David Lloyd, in his *Memoirs* (1688), speaks of his prudence, meekness, simplicity, patience and contentment; how he owed his health to his temperance and his devotion to Archery; and how he was "as facetious in Greek and Latine as he was solid in Arithmetique, Astronomy, and the Sphere of all Measures, Musick, etc." The following letter from the Ashmolean MSS., which is not given in the volume under notice, will show, as David Lloyd would say, the exactness of his style as of his judgment. It is published in J. O. Halliwell's *Letters on Scientific Subjects* (p. 93, 1841), and is dated Albury, Dec. 19th, 1652:

"Good Mr. Greatorix,—Give leave to intreate you to remember my service to my good friend Mr. Lilly, and to enquire of him and other astronomers about London, what they have observed concerning a comet, *stella crinita non caudata*, now and for 11 nightes together as often as the skie was cleere, in manner as I shall describe. Upon Thursday, Dec: 9th, I first saw it, neere the East, about 7 of the clock at night, a round dim light, about 4 degrees

* "To his worthy friend Master Evelyn, upon his translation of Lucretius."

† Hollar painted, in 1666, the portrait of "Elias Allen the Mathematician." It would be interesting to learn something of Elias.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

December, 1917.

Diagram for Calculations. By H. G. LLOYD. One card. 2s. 6d. net. 1917. (E. & F. N. Spon.)

Recreations in Mathematics. By H. E. LICKS. Pp. 155. $1·25. 1917. (D. van Nostrand, New York.)

Elementi di Aritmetica; con note storiche e numerose questione varie. Per le scuole medie superiori. Part I. *Numeri interi. Operazioni, divisibilità, numeri primi.* Fourth edition. By G. FAZZARI. Pp. 132. 1·70 L. 1918. (Trimarchi, Palermo.)

Elliptic Integrals. By HARRIS HANCOCK. Pp. 104. 6s. net. 1917. (Chapman, Hall.)

Bulletin of the American Mathematical Society.

Oct. 1917.

Integrals of Lebesgue. Pp. 1-47. G. A. BLISS.

Nov. 1917.

Irrational Transformations of the General Elliptic Element. Pp. 74-76. F. H. SAFFORD. *Note on the Parametric Representation of an Arbitrary Continuous Curve.* Pp. 77-82. D. JACKSON. *John Wallis as a Cryptographer.* Pp. 82-96. D. E. SMITH.

The Journal of the Indian Mathematical Society.

Oct. 1917.

Infinite Series and Arithmetical Functions. Pp. 174-186. F. HALLBERG. *Asymptotic Expansions of Integral Functions.* Pp. 186-201. K. B. MADHAVA.

of estimacioun from the head of the staire under the foot of Orion westward; the diameter of it seemed to be 16 inches; upon Saturday at 11 at night it was ascended nere the shild of Orion, almost as a right line through both the shoulders, so that the comet and they were about an equal distance asunder; then it went upward through the nose of the bull, till upon Tuesday night it was close, but a very small deale westward to the lowest star of the Pleiades. Upon Wednesday night it was come almost as high as the foote of Perseus, and upon Thursday night it was seene above the wing of that foote of Perseus westward higher then in a right line, through the star in that foote and the wing, and almost at their distance. The present Sunday, while I am writing, it is come within lesse than a degree of Gorgon's eye, and ½ a degree to the east side. The motoun decresethe nightly, and so doth the diameter, especially for these latter dayes, being now become but in shew not past halfe so big as at the first appearance. You shall doe me a favour to write or procure to be written to me what hath byne observed by the astronomers about this celestiall appearance, and what judgement they give of it,—your very loving friend, WILLIAM OUGHTRED."

Captain Ralph Greatorex (or Greatrex) was a noted mathematical instrument-maker of the time, and was known to many contemporaries of scientific interests, *e.g.* Evelyn, visiting Dr. Wilkins four years later at Whitehall, finds there Sir P. Neale, "famous for his optic glasses," and "Greatorix the mathematical instrument maker shew'd me his excellent invention to quench fire." The above letter written by a man on the verge of fourscore years shows no abatement of intellectual interest. The comet in question was one which attracted the attention of Newton, owing to the statement of Helvelius that it steadily enlarged as it receded from the sun in the proportion of 1 to 24, being almost equal to the sun in "absolute" magnitude when on the point of disappearance.* He also said that at one period it resembled the moon when half full, shining with a pale and dismal light, and commented on its caudal undulations. Such a notable discrepancy as this between the observations of two men of such repute is worthy of note, though no doubt it has not escaped the attention of astronomers. I find no reference to it in Chambers' *Story of the Comets* (Oxford, Clar. Press, 1909). As the examinee remarked: "Comets are very useful to give people things to talk about, and we should get on very badly without comets."

It may be easy to find similar instances of intellectual activity in matters of science in men of such a great age. At the moment we can think of but one, and that is even more striking. (Sir) J. F. Pollock (Senior Wrangler of 1807) wrote to his son in 1868: "My last paper (on the Theory of Numbers) . . . is to appear in the *Philosophical Transactions*. This is very gratifying; I wrote it since I was eighty-four, and I doubt whether there be any production of so ancient a person in the whole series."

But stay,—place aux dames! One of the early "suffragettes," to use a horrid word, published her *Molecular and Microscopic Science* when she was 88. Mary Somerville, the lady in question, remarked in later years: "In writing this book I made a great mistake, and repent it. Mathematics are the natural bent of my mind." So we find her up to the day of her death, at the age of 92, revising and completing a treatise, written years before, on the *Theory of Differences*, and studying Tait's *Quaternions*, which, with Salmon's *Higher Algebra*, had been sent her by Spottiswoode.

A chapter might be written on the recreations of old age. Caroline Herschel, between 88 and 89, delighted her circle by her great *feat*—of scratching her ear with her foot! The English take their pleasures sadly; who could rival the far-famed comet-sweeper in her merry moods?

Students of the history of our subject should be grateful to Prof. Cajori for his very clear and succinct account of so striking a figure as Oughtred, and those to whom most of the facts he produces are not unfamiliar will be glad to see his chapters on the influence of that great teacher upon mathematical progress and pedagogy, and none the less since they come from the pen of one who has consistently championed the claims of Oughtred to be

---

* *v.* Grant, *Hist. of Phys. Astr.* p. 301.

the inventor of the slide-rule.* Long ago De Morgan wrote: "the truth is that Oughtred invented it," and, in the *Penny Cyclopaedia* (Slide Rule), he again affirms that Oughtred was the inventor of the slide and the first who wrote about it, incidentally, by the way, throwing some interesting light on the way in which history can be written. It is a pity that Prof. Cajori, after his contrast between the views of Delamain and Oughtred as to the place of the slide-rule in the class-room, did not give us some impression as to the feeling of American teachers in the matter. Does the successful mechanical use of the instrument tend to destroy the intellectual interest in theory? Does it follow that the attendant at the switchboard is by the fact of his occupation content to go on without satisfying his curiosity as to the general principles of electrical science—*i.e.* is his curiosity stifled by a familiarity that breeds contempt? Or can the mastery of theory be best attained, as Delamain held, by the use of the instrument *pari passu* with the study of the theory? †

To the list of books which Newton read in early days at Cambridge should perhaps be added Wallis's *Arithmetic of Infinites*, which he had borrowed in the beginning of 1663. Brewster tells us that Newton's commonplace book, dated on p. 7, Jan, 1663-4, contained annotations therefrom. Prof. Cajori would please some of us better were he consistent in his reproductions in the matter of the original spelling, as he sometimes changes it and sometimes leaves it unaltered. It seems to us that we lose much of the picturesqueness and of the atmosphere, if we have a sentence like "I pray let me be remembered, though unknown, to Mistress Oughtred," foisted on us for: "I praie let me bee remembred tho' unknowne to Mrs. Owtrede," in which, moreover, we learn something of the variations in the spelling of a proper name, and of the indiscriminate use of Mrs. and Mistress, with which it is well for the young reader to become acquainted. On p. 19 it might have been stated that Mrs. Lichfield was not only a publisher, but the publisher of the 1652 edition. We seem to remember that she was no more pliable in her dealings with authors than some of those whose unpleasantness led the poet Campbell to the reflective remark that "Barabbas was a publisher," and to admire Buonaparte because he shot the unfortunate "bookseller" Palm. On p. 21 Prof. Cajori gives the date of the first use of the decimal point as 1616; he must reconsider this claim for Napier as against Pitiscus, whose *Canon Triangulorum* . . . was published in 1612, and is a work with which the great inventor of logarithms was probably familiar.

The author mentions the title "Oughtred Eatonensis" (? Oughtredus), and quotes from the *Budget of Paradoxes*: "He is an animal of extinct race, an Eton mathematician. Few Eton men, even of the minority which knows what a sliding rule is, are aware that the inventor was of their own school and college." Out of curiosity we recently applied to a distinguished Etonian for a list of Eton mathematicians, thinking from the preceding statement that the compilation of such a list would make no unwarrantable demands on his leisure. He replied that he had no time at his disposal for such research, and we are still doubtful as to what inference is to be drawn from that answer. J. F. W. Herschel, Etonian, was Senior Wrangler in 1813. And yet, as long as Eton men went to King's, and had to read some little mathematics as a condition for entry for the classical tripos, there must have been some men of preeminent ability who could hardly have failed to come out well in the lists of those days. De Morgan reminds us that the "Wooden Spoon" was frequently

---

* *Colorado College Publications.* Gen. Ser. 47 (1910), 84 (1915); *Nature*, 1909, vol. 82, p. 267; *History of Logarithmic Slide Rule*, 1909, p. 14 (N. York).

† "Practice multiplying simple numbers; ask nobody to help you, and you will rapidly get familiar with and fond of the slide rule .... Instructions are of no use. Find all this out for yourself."—Perry, *Practical Mathematics*, 1907, p. 11. "Much the best way of beginning to use the sliding rule is not by working given questions, but by setting the slide at hazard, and learning to read the questions which are thus fortuitously worked."—De Morgan, *Slide, or Sliding Rule*, P.C. xxii. (1842).

The appeal to hazard is not new. Recorde took the random replies of "suche children and ydeotes as happened to be in the place," and astounded his friends by getting right answers.—De Morgan, *Recorde*, P.C. xix. p. 331.

a distinguished classic, who wished merely to qualify himself for admission to the classical examination. And even at Oxford, where the subject was voluntary, Peel had a first class to himself 24 years before Gladstone's name was one of five in the same class. In Gladstone's diary we find cryptic utterances such as: "May, 16th. Sleepy. Mathematics, few and shuffling, and lecture," and then, again, a statement which is clear enough: "Every day I read, I am more and more thoroughly convinced of my incapacity for the subject." His examination lasted four days, and the excitement natural to the occasion was allayed as far as possible by readings from Wordsworth—and by daily draughts from the doctor.

Hobbes in his pretty, playful way taunted Wallis with having caught the "scab of symbols" from Oughtred, and Prof. Cajori quotes from a letter to John Collins from Wallis apropos of the *Clavis*: "It is true, that as in other things so in mathematics, fashions will daily alter, and that which Mr. Oughtred designed by great letters may be now by others designed by small; but a mathematician will, with the same ease and advantage, understand A_c and a^3 or aaa...." The curious incapacity to appreciate the value of symbols, and dislike of the printed page "horrent with mysterious spiculae," are matters of considerable psychological interest. At present we can run up the whole gamut of intellectual ability from those who read, perhaps with more pleasure than the poems of Wordsworth, the thousands of lines in the *Principia Mathematica*, to the student who persuasively urged upon his master that perhaps after all it is an unnecessary refinement to insist upon the essential difference between $2a$ and a^2.

We now come to a few slips, etc., in these pages. The prefix of R. P. and again of M. to names with and without initials on pp. 74 and 77 seems to require some explanation; it looks as if a little group of names were copied from an eighteenth century list, though Clairaut is as much entitled to "Monsieur" as any of his companions, and de L'Hospital is deprived both of his title and his "de" in favour of the more democratic prefix. For *mésure*, p. 77 note 2, read *mesure*; for *ouverto fibre elastiche* read *ovvero fibre elastiche*; for *conte* read *Conte*; for *Riccato* read *Riccati*; and on p. 79 for *Bassoni* should we not have *Bassano*, and *sublimioris* for *sublimoris*? For *Novae*, note 4 p. 80, read *Nova*; for *Pars*, note 7 p. 82, read *Paris*; and, in the same note, is *Specima* a classical word? The name Oswald Schresgensuchs, p. 83, is unfamiliar to us, and probably refers to Erasmus Osualdus Schrekhenfuchsius, or Schreckenfuss.

Apropos of the complaints of the obscurity of Oughtred's style, it might be worth recording that Newton says in his *Method of Fluxions* that he found the methods of the learned William so tedious that he was forced to set to work to invent an easier one.

We must not forget to draw the attention of readers who do not regularly see the *Q. J. of Mathematics*, that in vol. 46 (1916) Mr. Glaisher identifies with Oughtred the anonymous contributor of a table of hyperbolic logarithms in an Appendix to Edward Wright's translation of Napier's *Descriptio* (1618).

But we must bring to a close these too discursive remarks for which the only justification is a love for this period in history, and the interest that has been aroused in the writer's mind by the most suggestive and valuable little monograph which is the fruit of Prof. Cajori's latest labours in the field of historical research, and of the spirit that informs the Apocryphal line:

"Let us now praise great men and our fathers that begat us."

The Combination of Observations. By D. Brunt. Pp. x + 220. 8s. net. 1917. (Cambridge Univ. Press.)

Mr. Brunt has succeeded in covering considerable ground within the limits of some two hundred pages, and his book will receive a hearty welcome from those who desire to make themselves so far acquainted with the theory of errors and the method of least squares as to be able to utilise in an intelligent manner the knowledge they have acquired when applying it to the special investigations in which they are interested. For, as he is careful to impress upon his readers: "it is in no way justifiable to regard Least Squares as a magical instrument applicable to all problems," and again: "it cannot

be too strongly insisted upon that the methods of Least Squares cannot in any way improve upon the actual observations." The volume opens with a discussion of the nature of errors of observation. Proofs of the Normal Error Law are given, including the Eddington generalisation of Hagen's Proof of the Gaussian Law. The reader is warned that the "final justification of the use of Gauss's Error Curve rests upon the fact that it works well in practice," and that "the normal law is to be regarded as *proved by experiment*, and *explained* by Hagen's hypothesis." From the case of the probable error for one unknown we proceed to the chapter on observations of different weights, and here perhaps a reference might be given to cases in which errors of weighting may with safety be ignored. The formation of normal equations is then considered, and the next two chapters are devoted to their solution, to checks in computation, etc., with many illustrative examples. After chapters on the adjustment of conditioned observations and a few sections on the rejection of observations, we approach alternatives to the normal law, the use of skew curves, and the work in curve-fitting and corrections associated with the names of Pearson and Sheppard respectively, out of which discussion arises the treatment of correlation in chapter x. Sufficient space is then devoted to harmonic analysis from the standpoint of least squares, and to the periodogram. References are given throughout to original sources, and the illustrative examples are drawn from astronomy, physics, biology, chemistry, etc. As the author is at the front it is not surprising that here and there a misprint is to be found, but fortunately nothing of importance has escaped "the unfailing vigilance of the Press."

Differential Calculus. By H. B. Phillips. Pp. vi+162. 5s. 6d. net. 1916. (Wiley & Sons; Chapman & Hall.)

We do not quite see why this book has been imported. It is interesting as an attempt to cover the essentials in one term so as to get on to the integral calculus as soon as possible. But its logical deficiencies are rather serious at times. The use of velocities, etc., at an early stage, now generally recognised as the most useful method of approach, is postponed to the middle of the book; the author's statement of Rolle's Theorem is accompanied by something that is fortunately rare in mathematical text-books—two illustrations that do not illustrate. We do not say that the book has not some good qualities, but they are not so novel, and do not so far outweigh the defects, as to justify its class introduction at the price of 5s. 6d. net, or its use as sufficiently instructive and suggestive to our average British teacher.

THE LIBRARY.

Change of Address.

The Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

A copy of Prof. A. N. Whitehead's *Organisation of Thought* has been added to the Library.

Scarce Back Numbers.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

ERRATUM.

P. 140, line 1, *for* OPB *read* OPC.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.